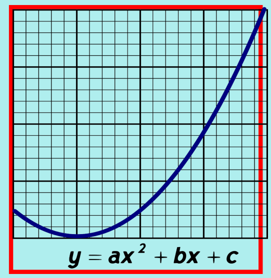
Math 25

Fall 2017

Lecture 11



Open notes Quiz

Given $2, 10, 18, 26, \ldots$

$+8 \quad +8 \quad +8$

1) find d

$\boxed{d=8}$

2) find a_{25}

$a_n = a_1 + (n-1)d$

$a_{25} = 2 + (25-1)8$

$= 2 + 200 - 8 = \boxed{194}$

3) find S_{100}

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$S_{100} = \frac{100}{2}[2(2) + (100-1)\cdot 8]$

$= 50[4 + 800 - 8]$

$= 50[796]$

$\boxed{S_{100} = 39{,}800}$

Binomial Expansion (Thrm)

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Find $(a+b)^n$

$a^n b^0 + a^{n-1} b^1 + a^{n-2} b^2 + \cdots + a^0 b^n$

Binomial coef.

$_nC_r$ n items (Different), choosing r of them

order does not matter.

$$_nC_r = \frac{n!}{r! \cdot (n-r)!}$$

$$_{10}C_7 = \frac{10!}{7! \cdot (10-7)!}$$

$$= \frac{10!}{7! \cdot 3!}$$

$$= \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8}^{4} \cdot \cancel{7!}}{\cancel{7!} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}$$

$$= \boxed{120}$$

$$_nC_r = \binom{n}{r}$$

find $\binom{8}{3} = {}_8C_3 = \frac{8!}{3! \cdot (8-3)!}$

$= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{3 \cdot 2 \cdot 1 \cdot \cancel{5!}} = \boxed{56}$

$(a+b)^n =$

$\binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots\cdots$

$\binom{n}{n} a^0 b^n$

Expand $(a+b)^6$

$= \binom{6}{0} a^6 b^0 + \binom{6}{1} a^5 b^1 + \binom{6}{2} a^4 b^2 + \binom{6}{3} a^3 b^3$

$+ \binom{6}{4} a^2 b^4 + \binom{6}{5} a^1 b^5 + \binom{6}{6} a^0 b^6$

$= 1a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4$

$+ 6ab^5 + 1b^6$

Find $\underbrace{\text{the 6th term}}$ of $(a+b)^{13}$

$$\binom{13}{5} a^8 b^5 = \boxed{1287\, a^8 b^5}$$

Find the $\underbrace{\text{5th term}}$ of $(2x+y^3)^{10}$

Let $a=2x$, $b=y^3$

$$(a+b)^{10} \rightarrow \binom{10}{4} a^6 b^4$$

$$= 210\cdot(2x)^6\cdot(y^3)^4 = 210\cdot 2^6 x^6 y^{12} = \boxed{13440x^6y^{12}}$$

Find the middle term of the expansion of

$$\left(3x^5 - \frac{2}{3}y^{10}\right)^8$$

9 terms

$(a+b)^n$ has $(n+1)$ terms.

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^{10} \rightarrow 11$ terms

$(a+b)^{25} \rightarrow 26$ terms

$\boxed{\text{4 terms}}$ $\boxed{\text{4 terms}}$

Middle term

5th term.

Find the middle term of the expansion of

$$\left(3x^5 - \frac{2}{3}y^{10}\right)^8$$

we want the 5th term.

$$(a+b)^8 \longrightarrow \binom{8}{4}a^4b^4$$

$a = 3x^5$, $b = \frac{-2}{3}y^{10}$

$$= 70\left(3x^5\right)^4\left(\frac{-2}{3}y^{10}\right)^4$$

$$= 70 \cdot \cancel{3^4} \cdot (x^5)^4 \cdot \frac{(-2)^4}{\cancel{3^4}} \cdot (y^{10})^4 = 70 \cdot 16\, x^{20}y^{40}$$

$$= \boxed{1120x^{20}y^{40}}$$

Infinite Geometric Series

$$10 + 5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \cdots$$

$a_1 = 10$, $r = \frac{1}{2}$

$$S_\infty = \frac{a_1}{1-r} \qquad S_\infty = \frac{10}{1-\frac{1}{2}} = \frac{10}{\frac{1}{2}}$$

$$\boxed{S_\infty = 20}$$

Find $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+1}$

$= \left(\frac{2}{3}\right)^{1+1} + \left(\frac{2}{3}\right)^{2+1} + \left(\frac{2}{3}\right)^{3+1} + \cdots$

$= \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{3} + \left(\frac{2}{3}\right)^{4} + \cdots$

$= \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$

$a_1 = \frac{4}{9}$

$r = \frac{2}{3}$

$S_\infty = \frac{a_1}{1-r} = \frac{4/9}{1-2/3} = \frac{\frac{4}{9}}{\frac{1}{3}}$

$= \boxed{\frac{4}{3}}$

Functions

Input $\rightarrow$ Domain

output $\rightarrow$ Range

Any input values, we cannot have more than one output Values.

$f(x) = x^2 - 4x$

$f(0) = 0^2 - 4(0) = 0$ $\qquad f(0) = 0$

$f(-3) = (-3)^2 - 4(-3) = 9 + 12 = 21$ $\qquad f(-3) = 21$

Some functions are one-to-one functions

Different inputs $\longrightarrow$ Different outputs

$f(x) = x^2$

$f(2) = 4$

$f(-2) = 4$

$\Longrightarrow$ Not one-to-one function

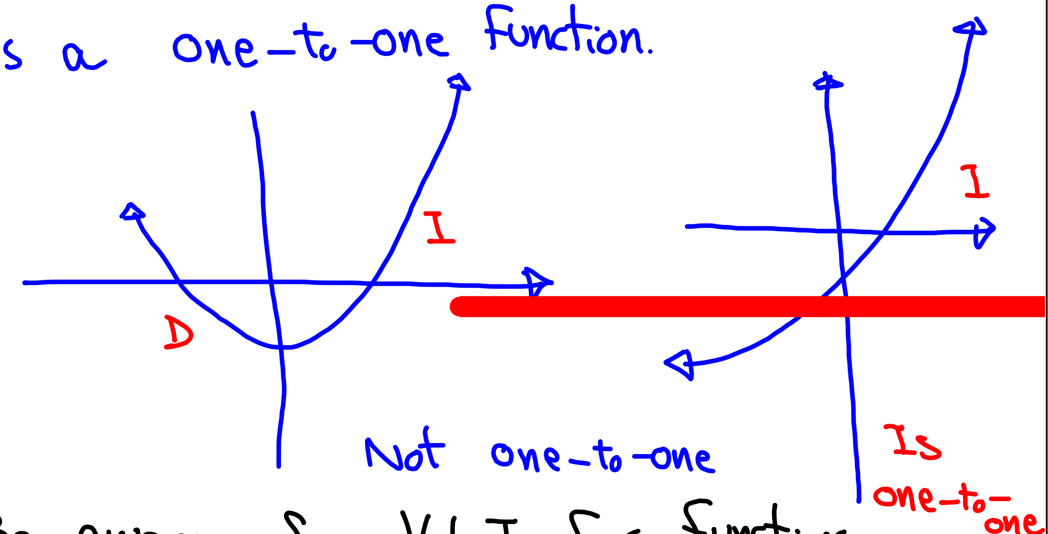
$f(x) = x^3$

$f(2) = 8$

$f(-2) = -8$

$\longrightarrow$ Is a one-to-one function.

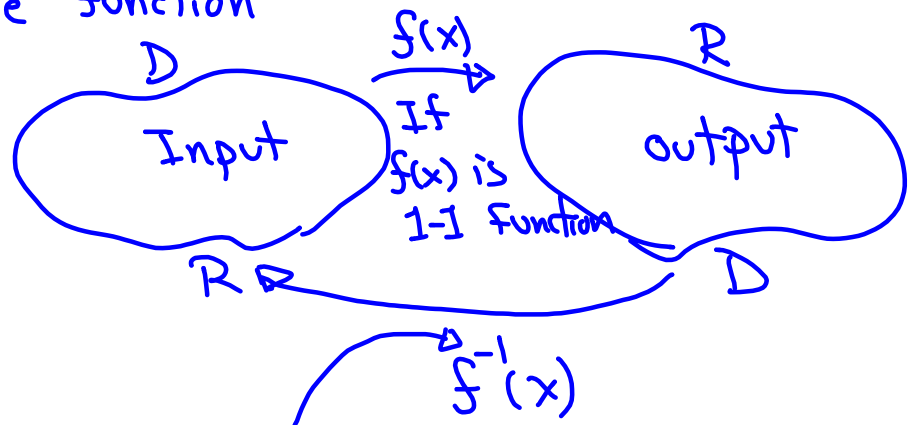
A function that is increasing all the way, or that is decreasing all the way is a one-to-one function.



Be aware of V.L.T. for functions

H.L.T. for 1-1 function

Any one-to-one function has an inverse function



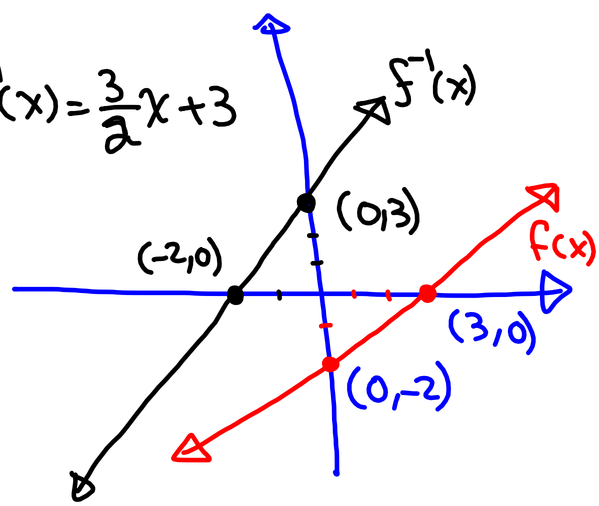
Not an exponent

Does not mean reciprocal

$f(x) = \frac{2}{3}x - 2$

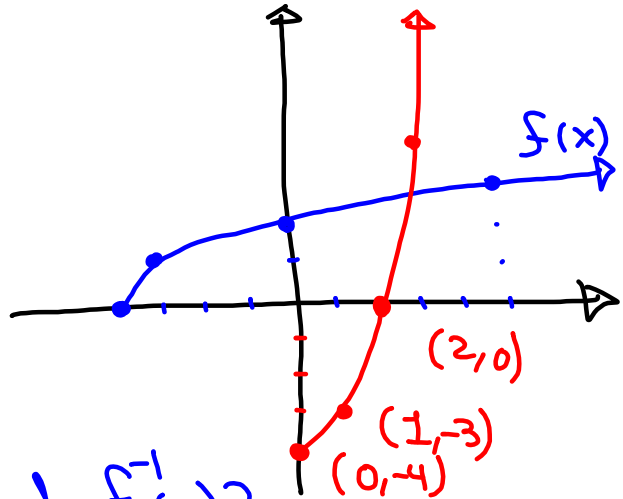
$y = \frac{2}{3}x - 2$

$f^{-1}(x) = \frac{3}{2}x + 3$



$f(x) = \sqrt{x+4}$

x	y
-4	0
-3	1
0	2
5	3



How to Find $f^{-1}(x)$?

1) Replace $f(x)$ with y

2) Switch x & y

3) Solve for y

4) Replace y with $f^{-1}(x)$

$f(x) = \sqrt{x+4}$

$y = \sqrt{x+4}$

$x = \sqrt{y+4}$

$x^2 = y+4$

$x^2 - 4 = y$

$f^{-1}(x) = x^2 - 4$

$x \geq 0$

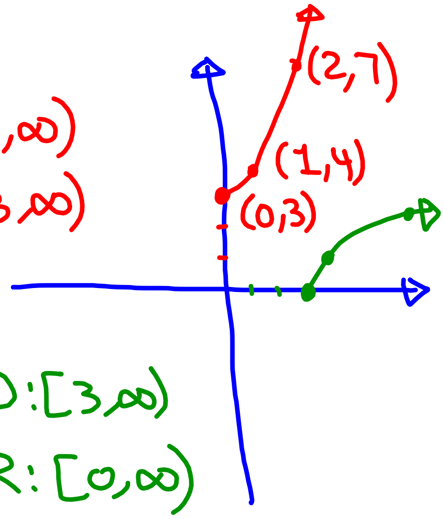
$f(x) = x^2 + 3$, $x \geq 0$

Graph $f(x)$

Discuss domain, range

D: $[0, \infty)$

R: $[3, \infty)$



Graph $f^{-1}(x)$

Discuss domain, range

D: $[3, \infty)$

R: $[0, \infty)$

Exponential Functions $f(x) = b^x$, $b > 0$, $b \neq 1$

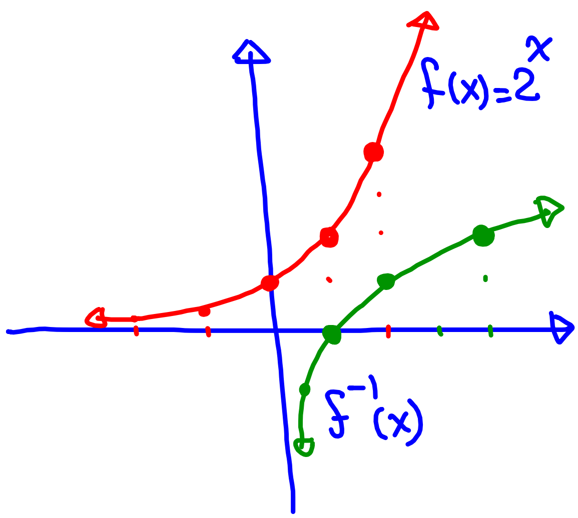
$f(x) = 2^x$

x	y
0	1
1	2
2	4
3	8

x	y
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$
-3	$\frac{1}{8}$

$x \to \infty$
$y \to \infty$

$x \to -\infty$
$y \to 0$



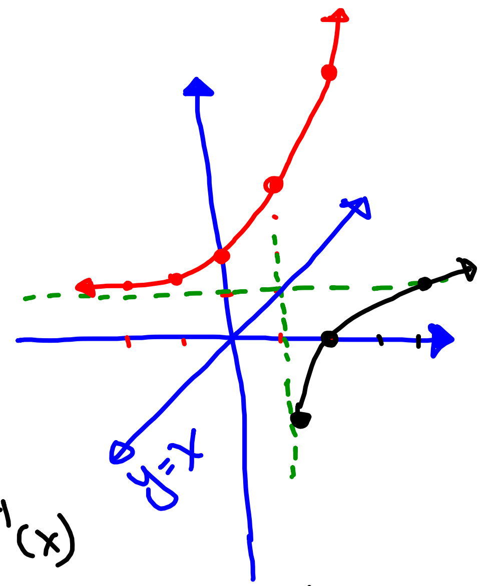
Graph $f(x) = 3^x + 1$

x	y
0	2
1	4
2	10

x	y
-1	$\frac{1}{3} + 1$
-2	$\frac{1}{9} + 1$
-3	$\frac{1}{27} + 1$

$x \to \infty$
$y \to \infty$

$x \to -\infty$
$y \to 1$



Graphs of $f(x)$ & $f^{-1}(x)$ are symmetric with respect to the line $y = x$

Exponential Equations

If $b^x = b^y$, $b > 0$, and $b \neq 1$, then $x = y$.

Solve $2^{3x-1} = 32$

$2^{3x-1} = 2^5$

$3x - 1 = 5$

$\boxed{x = 2}$

Solve $25^{3x+4} = 125^{x-5}$

$(5^2)^{3x+4} = (5^3)^{x-5}$

$5^{2(3x+4)} = 5^{3(x-5)}$

$2(3x+4) = 3(x-5)$

$6x + 8 = 3x - 15$

$3x = -23$

$x = -23/3$

$\left\{ \frac{-23}{3} \right\}$

Solve $3^x \cdot 3^{x-2} = \frac{1}{27}$

$$3^{x+x-2} = \frac{1}{3^3}$$

$$3^{2x-2} = 3^{-3}$$

$$2x-2=-3$$

$$2x=-1$$

$$\boxed{x=\frac{-1}{2}}$$

$\left\{\frac{-1}{2}\right\}$

Solve

$$7^{2x-3} = \left(\frac{1}{49}\right)^{x+1}$$

$$7^{2x-3} = (7^{-2})^{x+1}$$

$$7^{2x-3} = 7^{-2(x+1)}$$

$$2x-3=-2(x+1)$$

$$\vdots$$

$$\boxed{x=\frac{1}{4}}$$

$\{1/4\}$

The inverse of exponential functions are logarithmic functions.

$f(x) = b^x$, $b>0$, $b\neq 1$

$f^{-1}(x) = \log_b x$, $x>0$

$F(x) = 2^x$ $\qquad$ $F^{-1}(x) = \log_2 x$

On your calc. log $\rightarrow$ Common log $\rightarrow$ $b=10$

ln $\rightarrow$ Natural log $\rightarrow$ $b=e$

find $\log 10 = 1$

find $\ln 25 \approx 3.22$

find $\dfrac{\log 12}{\log 2} \approx 3.58$

find $\dfrac{\ln 12}{\ln 2} \approx 3.58$

when Solving $b^x = y$

$$\text{Exponent} = \frac{\log \text{RHS}}{\log \text{base}}$$

$3^x = 200$ $\quad \boxed{x = \dfrac{\log 200}{\log 3}}$ $\quad \boxed{x \approx 4.82}$

Exact Ans. Rounded to 2-decimal

$5^{x-2} = 2017$

$$\text{Exponent} = \frac{\log \text{RHS}}{\log \text{Base}}$$

$x - 2 = \dfrac{\log 2017}{\log 5}$ $\quad \boxed{x \approx 6.73}$

$x = \dfrac{\log 2017}{\log 5} + 2$

Solve $7^{2x+5} = 2345$

$$2x+5 = \frac{\log 2345}{\log 7}$$

$$2x = \frac{\log 2345}{\log 7} - 5$$

$$\boxed{x = \frac{1}{2}\left[\frac{\log 2345}{\log 7} - 5\right]}$$

$$\boxed{x \approx -.51}$$

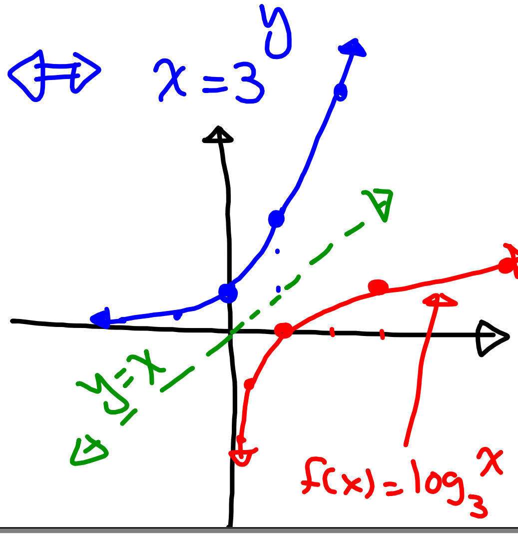
Graphing $f(x) = \log_b x$, $x > 0$
$b > 0, b \neq 1$

$f(x) = \log_b x \Rightarrow y = \log_b x \Leftrightarrow x = b^y$

Graph $f(x) = \log_3 x \Leftrightarrow x = 3^y$

x	y
1	0
3	1
9	2

x	y
1/3	-1
1/9	-2

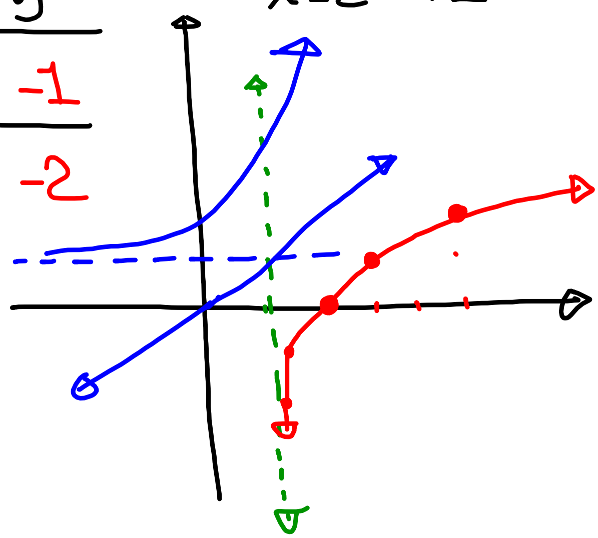


Graph $f(x)=\log_2(x-1) \Leftrightarrow x-1=2^y$

$x=2^y+1$

x	y
2	0
3	1
5	2

x	y
$1+\frac{1}{2}$	-1
$1+\frac{1}{4}$	-2



Graph $f(x)=\log_4(x+2) \Leftrightarrow x+2=4^y$

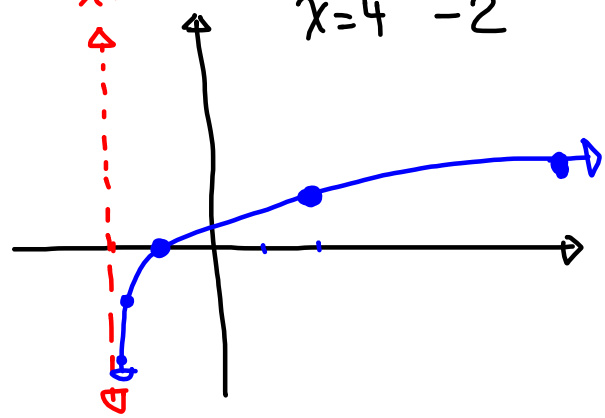
$x=4^y-2$

x	y
-1	0
2	1
14	2

x	y
$-2+\frac{1}{4}$	-1
$-2+\frac{1}{16}$	-2

$x \to -2$

$x=-2$



Solving simple logarithmic equations

$5 = \log_2(x-3) \Leftrightarrow x-3 = 2^5$

$x-3 = 32$

$\boxed{x=35}$ $\{35\}$

$2 = \log_5(3x-1) \Leftrightarrow 5^2 = 3x-1$

$25 = 3x-1$

$\left\{\frac{26}{3}\right\}$

$26 = 3x$ $\boxed{x=\frac{26}{3}}$

Solve $\log_2(x^2-4x) = 5$

$x^2-4x = 2^5$

$x^2-4x-32 = 0$

$(x-8)(x+4) = 0$

$x=8$ $x=-4$ $\{8,-4\}$

More practice with M.I.

Mathematical Induction

Prove by M.I. that

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$$

$a_n =$ nth term is 2^{n-1}

$a_{n+1} = (n+1)$th term is $2^{n+1-1} = 2^n$

	LHS		RHS
$n=1$	1	$=$	$2^1 - 1 = 2 - 1 = 1$ ✓
$n=2$	$1+2$	$=$	$2^2 - 1 = 4 - 1 = 3$ ✓
$n=3$	$1+2+2^2$	$=$	$2^3 - 1 = 8 - 1 = 7$ ✓

Assume it works for $n=k$

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} = 2^k - 1$$

Add the next term to both sides

$$\underbrace{1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} + 2^k}_{\text{How terms on the LHS? } (k+1) \text{ terms}} = 2^k - 1 + 2^k$$

$$= 2 \cdot 2^k - 1$$

$$= 2^{k+1} - 1$$

Prove by M.I.

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

	LHS		RHS
$n=1$	$1\cdot 2$	$=$	$\frac{1(2)(3)}{3}$ ✓
$n=2$	$1\cdot 2 + 2\cdot 3$	$=$	$\frac{2(3)(4)}{3}$
	$2+6$		
	8	$= 8$ ✓	

$$= \frac{(k+1)(k+1+1)(k+1+2)}{3}$$

Assume it works for $n=k$

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

Add the next term to both sides

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k(k+1) + (k+1)(k+2) =$$

$$\frac{k(k+1)(k+2)}{3} + \frac{(k+1)(k+2)\cdot 3}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

On the LHS, we have $(k+1)$ terms

$$= \frac{(k+1)(k+1+1)(k+1+2)}{3}$$

Open-notes Quiz

Consider $3, 6, 12, 24, \ldots$

find

1) r

2) a_{10}

3) S_{10}